

ON ASYMPTOTIC PROPERTY C

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ABSTRACT. The notion of asymptotic property C was introduced in [D1]. We show that asymptotic property C is preserved by finite products. We also show that countable restricted direct products of countable groups with finite asymptotic dimension have asymptotic property C. Then we introduce hyperbolic property C, an infinite dimensional version of hyperbolic dimension.

1. INTRODUCTION

Asymptotic dimension was introduced by Mikhail Gromov in [G] to study finitely-generated groups, but this notion applies to all metric spaces. In [Yu1] Guoliang Yu proved the Novikov higher signature conjecture for groups with finite asymptotic dimension. In [Yu2] Yu introduced property A, a dimension-like property weaker than finite asymptotic dimension, and proved the coarse Baum-Connes conjecture for groups with property A. This stimulated interest in large scale dimension-like properties which are satisfied by spaces with infinite asymptotic dimension and which imply property A. Such infinite dimensional properties include asymptotic property C [D1], finite decomposition complexity [GT], and straight finite decomposition complexity [DZ]. A basic question is which constructions preserve these properties. In section 3 we show that asymptotic property C is preserved by finite direct products, answering a question from [BBGRZ] and [BM]. We also show that asymptotic property C holds for countable restricted direct products of groups with finite asymptotic dimension. This can be viewed as a partial affirmative answer to question 3.2 from [Y].

In [DZ] the following implications were demonstrated for metric spaces:

$$\text{asymptotic property C} \implies \text{sFDC} \implies \text{property A}$$

In section 4 we introduce the notions of hyperbolic property C and weak hyperbolic property C, which are dimension-like properties for spaces with infinite hyperbolic dimension. These satisfy the following implications for metric spaces:

$$\begin{aligned} \text{asymptotic property C} &\implies \text{hyperbolic property C} \implies \text{weak hyperbolic property C} \\ &\implies \text{sFDC} \implies \text{property A} \end{aligned}$$

The author has been made aware that the preservation of asymptotic property C by finite direct products was shown independently in [BN]. The author would like to thank Alexander Dranishnikov for many stimulating conversations during the writing of this paper.

2. PRELIMINARIES

Let X be a metric space. For nonempty $A, B \subset X$, we let $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

Let $R > 0$. A family \mathcal{U} of nonempty subsets of X is R -disjoint if $d(A, B) > R$ for all $A, B \in \mathcal{U}$ with $A \neq B$.

A family \mathcal{U} of subsets of X is uniformly bounded if $\text{mesh } \mathcal{U} = \sup\{\text{diam}(U) : U \in \mathcal{U}\} < \infty$.

The asymptotic dimension of X does not exceed n , written $\text{asdim } X \leq n$, if for every $R > 0$ there are uniformly bounded R -disjoint families $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a cover of X . We say X has finite asymptotic dimension if there exists n with $\text{asdim } X \leq n$.

For a family of metric spaces $\{X_\alpha\}$, we say $\text{asdim } \{X_\alpha\} \leq n$ if for every $R > 0$ there exists $D \geq 0$ such that for every α there exist R -disjoint families $\mathcal{U}_0, \dots, \mathcal{U}_n$ such that $\text{mesh } \mathcal{U}_i \leq D$ for $0 \leq i \leq n$ and $\bigcup_{i=0}^n \mathcal{U}_i$ is a cover of X_α .

We recall the definition of asymptotic property C from [D1].

Definition 2.1. A metric space X has *asymptotic property C* if for every sequence $R_0 \leq R_1 \leq R_2 \dots$ of positive reals there exists $n \geq 0$ and uniformly bounded R_i -disjoint families \mathcal{U}_i , $i = 0, \dots, n$ such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a cover of X .

For metric spaces X, Y , a function $f : X \rightarrow Y$ is a coarse embedding if there exist nondecreasing functions $\rho_-, \rho_+ : [0, +\infty)$ with

$$\lim_{t \rightarrow +\infty} \rho_-(t) = \lim_{t \rightarrow +\infty} \rho_+(t) = +\infty$$

and

$$\rho_-(d(x, y)) \leq d(f(x), f(y)) \leq \rho_+(d(x, y))$$

for all $x, y \in X$.

The following is easy to check.

Proposition 2.2. For metric spaces X, Y with a coarse embedding $f : X \rightarrow Y$, if Y has asymptotic property C , then X has asymptotic property C .

Given a sequence of pointed metric spaces $(X_i, x_i)_{i=1}^\infty$, we define its restricted product $\times_{i=1}^\infty X_i$ to be the set of sequences $(a_i)_{i=1}^\infty$ with each $a_i \in X_i$ and $a_i = x_i$ for all but finitely many i . We equip $\times_{i=1}^\infty X_i$ with the metric

$$d((a_i), (b_i)) = \sum_{i=1}^\infty i \cdot d(a_i, b_i)$$

In [S] it was shown that any two proper, left-invariant metrics on a countable group are coarse equivalent. Given a sequence of countable groups $(G_i)_{i=1}^\infty$ each equipped with a proper, left-invariant metric, we choose the identity as the base point of each G_i . Then the restricted product $\times_{i=1}^\infty G_i$ is a countable group with the operation which applies the group operation of each G_i coordinate-wise, and clearly the metric defined above is proper and left-invariant.

A tree is a connected acyclic graph. We view a tree T as a metric space by equipping its set of vertices with the shortest path metric, i.e. for $u, v \in T$ we say $d(u, v)$ is the length of the shortest path in T from u to v .

For a tree T with root e and $0 \leq m < n$, we define the annulus from m to n :

$$[m, n) = \{v \in T : m \leq d(v, e) < n\}$$

Let \mathcal{X}, \mathcal{Y} be metric families and $R > 0$. We say \mathcal{X} is *R-decomposable over* \mathcal{Y} , denoted $\mathcal{X} \xrightarrow{R} \mathcal{Y}$, if for any $X \in \mathcal{X}$, $X = \bigcup(\mathcal{U}_1 \cup \mathcal{U}_2)$, where \mathcal{U}_1 and \mathcal{U}_2 are R -disjoint families such that $\mathcal{U}_1 \cup \mathcal{U}_2 \subset \mathcal{Y}$. We recall the definition of straight finite decomposition complexity from [DZ].

Definition 2.3. A metric family \mathcal{X} has *straight finite decomposition complexity* (sFDC) if for every sequence $R_0 \leq R_1 \leq \dots$ of positive reals there exists $n \in \mathbb{N}$ and metric families \mathcal{U}_i , $i = 0, \dots, n$ such that \mathcal{U}_n is uniformly bounded and

$$\mathcal{X} \xrightarrow{R_0} \mathcal{U}_0 \xrightarrow{R_1} \mathcal{U}_1 \xrightarrow{R_2} \dots \xrightarrow{R_n} \mathcal{U}_n$$

For a metric space X , $r > 0$ and $x \in X$, let $B_r(x)$ denote the open ball in X of radius r centered at x .

A subset $U \subset X$ of a metric space X is (N, R) -large scale doubling if for every $x \in X$ and every $r \geq R$, $B_{2r}(x) \cap U$ can be covered by N balls of radius r with centers in X .

A family \mathcal{U} of subsets of a metric space X is *uniformly large scale doubling* if there exists (N, R) such that each $U \in \mathcal{U}$ is (N, R) -large scale doubling, and such that every finite union of sets in \mathcal{U} is (N, R') -large scale doubling for some $R' > 0$.

A family \mathcal{U} of subsets of a metric space X is *weakly uniformly large scale doubling* if there exists (N, R) such that each $U \in \mathcal{U}$ is (N, R) -large scale doubling.

3. ASYMPTOTIC PROPERTY C

Theorem 3.1. *If X and Y are metric spaces with asymptotic property C, then $X \times Y$ has asymptotic property C.*

Proof. Fix a sequence R_1, R_2, \dots of positive real numbers. Use a bijection $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ to reindex this sequence as $\{R_{m,n} : m, n \in \mathbb{N}\}$. Since Y has asymptotic property C, for each m there is $n(m) \in \mathbb{N}$ such that there exist $\mathcal{V}_1^m, \mathcal{V}_2^m, \dots, \mathcal{V}_{n(m)}^m$ collections of subsets of Y such that each \mathcal{V}_n^m is $R_{m,n}$ -disjoint and uniformly bounded, and

$$\bigcup_{n=1}^{n(m)} \mathcal{V}_n^m$$

is a cover of Y . For each m , let $R'_m = \max(R_{m,1}, R_{m,2}, \dots, R_{m,n(m)})$

Now we apply the asymptotic property C of X to the sequence R'_1, R'_2, \dots to obtain a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_k$ of collections of subsets of X such that each \mathcal{U}_m is R'_m -disjoint and uniformly bounded, and

$$\bigcup_{m=1}^k \mathcal{U}_m$$

is a cover of X .

For each $m \leq k$ and $n \leq n(m)$, let

$$\mathcal{W}_{m,n} = \{U \times V : U \in \mathcal{U}_m \text{ and } V \in \mathcal{V}_n^m\}$$

Clearly each $\mathcal{W}_{m,n}$ is uniformly bounded and $R_{m,n}$ -disjoint, and

$$\bigcup_{m \leq k, n \leq n(m)} \mathcal{W}_{m,n}$$

is a cover of $X \times Y$. By reindexing using f^{-1} , we get indices $n_1 < n_2 < \dots < n_l$ and families $\mathcal{W}_{n_1}, \mathcal{W}_{n_2}, \dots, \mathcal{W}_{n_l}$. For each $n < n_l$ with $n \neq n_i$ for all $i < l$, let $\mathcal{W}_n = \emptyset$. Then each \mathcal{W}_n is R_n -disjoint and uniformly bounded, and

$$\bigcup_{j=1}^{n_l} \mathcal{W}_j$$

is a cover of $X \times Y$. We conclude that $X \times Y$ has asymptotic property C . \square

Lemma 3.2. *For any $R > 0$ and any R -disjoint family \mathcal{V} of annuli with uniformly bounded width in a tree T , \mathcal{V} has a uniformly bounded R -disjoint refinement.*

Proof. Let $w = \sup\{\text{width}(V) : V \in \mathcal{V}\}$. The family of annuli in \mathcal{V} distance at most R from e is uniformly bounded, so it is enough to consider the annuli which are distance $> R$ from e . So let $[a, b) = I \in \mathcal{V}$ with $a > R$. For $x, y \in I$, say $x E y$ if $(x|y)_e \geq a - R$. Note that E is an equivalence relation since T is a tree. If $x E y$, then

$$d(x, y) = d(x, e) + d(y, e) - 2(x|y)_e < 2b - 2(a - R) = 2b - 2a + 2R \leq 2w + 2R$$

On the other hand, if x, y are not E -related, then

$$d(x, y) = d(x, e) + d(y, e) - 2(x|y)_e > 2a - 2(a - R) = 2R$$

Hence the E -classes of I are uniformly bounded and R -disjoint. \square

Given a tree T and an R -disjoint family of annuli \mathcal{V} in T , we denote the refinement obtained from the above lemma by $\text{Ref}_R(\mathcal{V})$. The following theorem makes use of an argument analogous to that of the main theorem of [Y].

Theorem 3.3. *If $(T_i)_{i=1}^\infty$ is a sequence of trees, then $\times_{i=1}^\infty T_i$ has asymptotic property C .*

Proof. Let $R_0 < R_1 < R_2 < \dots$ be arbitrary positive reals. Choose $k, m \in \mathbb{N}$ such that $R_0 < k$ and $R_{k2^k} < m$. Each T_i has asymptotic dimension at most 1, so we have bounded and R_{k2^k} -disjoint families $\mathcal{V}_0^i, \mathcal{V}_1^i$ for $1 \leq i \leq k + m$ such that $\mathcal{V}_0^i \cup \mathcal{V}_1^i$ covers T_i . For each i , replace each set in \mathcal{V}_1^i with its intersection with the complement of $\cup \mathcal{V}_0^i$ so we have $(\cup \mathcal{V}_0^i) \cap (\cup \mathcal{V}_1^i) = \emptyset$. Note that $\times_{i=1}^\infty T_i$ admits a coarse embedding into a restricted product of trees each with at least 2 vertices, so we may assume that each V_0^i and each V_1^i is nonempty.

For each $i \in \mathbb{N}$, let $[a, b)_i$ denote the annulus in T_i from a to b . Let $S = R_0 + R_{k2^k}$. Fix a bijection $\psi : \{1, 2, \dots, 2^m\} \rightarrow \{0, 1\}^m$. For each $i \in \{1, 2, \dots, k\}$ and $l \in \{1, 2, \dots, 2^m\}$, let

$$C_l^i = \text{Ref}_{R_0}(\{[0, (2^m + l)S - R_0)_i\} \cup \{(2^m n + l)S, (2^m(n+1) + l)S - R_0)_i : n \in \mathbb{N}\})$$

$$D_l^i = \text{Ref}_{R_{k2^k}}\{[(2^m n + l)S - R_0, (2^m n + l)S)_i : n \in \mathbb{N}\}$$

$$W_l = \left\{ \prod_{i=k+1}^{k+m} V_i : V_i \in \mathcal{V}_{\psi(l)_{i-k}}^i \right\}$$

Note that for each i and l , C_l^i is R_0 -disjoint and $C_l^i \cup D_l^i$ covers T_i . Also, for each i we have that $\bigcup_{l=1}^{2^m} D_l^i$ is R_{k2^k} -disjoint, each W_l is R_{k2^k} -disjoint, $\bigcup_{l=1}^{2^m} W_l$ is disjoint, and $\bigcup_{l=1}^{2^m} W_l$ covers $\prod_{i=k+1}^{k+m} T_i$.

Now fix a bijection $\varphi : \{1, 2, \dots, 2^k\} \rightarrow \{0, 1\}^k$, let

$$\mathcal{U}_0 = \left\{ \prod_{i=1}^k C_i \times W \times \prod_{i>k+m} \{x_i\} : (C_1, C_2, \dots, C_k, W) \in \bigcup_{l=1}^{2^m} \left(\prod_{i=1}^k C_l^i \times W_l \right), (x_i) \in \times_{i=k+m+1}^\infty T_i \right\}$$

For each $s \in \{1, 2, \dots, k\}$ and $t \in \{1, 2, \dots, 2^k\}$ let

$$\mathcal{U}_{2^k(s-1)+t} = \left\{ \prod_{i=1}^{s-1} V_i \times D \times \prod_{i=s+1}^k V_i \times W \times \prod_{i>k+m} \{x_i\} : \right. \\ \left. V_i \in \mathcal{V}_{\varphi(t)_i}^i, i \in \{1, \dots, s-1, s+1, \dots, k\}, (D, W) \in \bigcup_{l=1}^{2^m} (D_l^s \times W_l), (x_i) \in \times_{i=k+m+1}^\infty T_i \right\}$$

The collection of all C_l^i, D_l^i , and W_l^i is uniformly bounded, so clearly the \mathcal{U}_j , $0 \leq j \leq k2^k$ are uniformly bounded.

To see that \mathcal{U}_0 is R_0 disjoint, let $U = C_1 \times C_2 \times \dots \times C_k \times W \times \prod_{i>k+m} \{x_i\}$ and $U' = C'_1 \times C'_2 \times \dots \times C'_k \times W' \times \prod_{i>k+m} \{x'_i\}$ be distinct sets in \mathcal{U}_0 , and let $a \in U, b \in U'$. If (x_i) and (x'_i) are distinct, then $a_i \neq b_i$ for some $i > k+m$, so $d((a_i), (b_i)) > k > R_0$. If W and W' are distinct, then they are disjoint, so $a_i \neq b_i$ for some $i > k$. Otherwise $W = W'$, so we have $W, W' \in W_l$ for some $1 \leq l \leq 2^m$. Hence $C_i, C'_i \in C_l^i$, but $C_i \neq C'_i$ for some i , so $d((a_i), (b_i)) > R_0$ since C_l^i is R_0 -disjoint for each i .

Each \mathcal{U}_j for $1 \leq j \leq k2^k$ can be shown to be R_{k2^k} -disjoint along similar lines. It is also not hard to see that the \mathcal{U}_j cover $\times_{i=1}^\infty T_i$. \square

Theorem 3.4. *Suppose $(G_i)_{i=1}^\infty$ is a sequence of countable groups each of finite asymptotic dimension. Then $\times_{i=1}^\infty G_i$ has asymptotic property C.*

Proof. By Theorem A from [D2], every countable group of asymptotic dimension n admits a coarse embedding into a product of $n+1$ trees equipped with the sup metric. Hence there exists a sequence of trees $(T_k)_{k=1}^\infty$ and a strictly increasing function $K : \mathbb{N} \rightarrow \mathbb{N}$ such that for each $i \in \mathbb{N}$ we have a coarse embedding $f_i : G_i \rightarrow \prod_{k=K(i)}^{K(i+1)-1} T_k$, where $\prod_{k=K(i)}^{K(i+1)-1} T_k$ is equipped with the sup metric. Since each G_i has bounded geometry, we may assume that each f_i is injective. For each i fix proper nondecreasing maps

$$\rho_-^i, \rho_+^i : [0, \infty) \rightarrow [0, \infty), \lim_{t \rightarrow \infty} \rho_-^i(t) = \infty$$

witnessing that f_i is a coarse embedding. We may assume for every $t > 0$ that

$\rho_+^i(t) \leq \rho_+^j(t)$ for all $i \leq j$. Let $F : \times_{i=1}^\infty X_i \rightarrow \times_{k=1}^\infty T_k$ be defined by

$$F((x_i)_{i=1}^\infty) = (f_i(x_i))_{i=1}^\infty$$

Note that each $f_i(x_i) \in \prod_{k=K(i)}^{K(i+1)-1} T_k$, so we view $(f_i(x_i))_{i=1}^\infty$ as an element of $\times_{i=1}^\infty T_i$ by appending the terms. Now let $x, y \in \times_{i=1}^\infty G_i$ and let $M = K(d(x, y) + 1)$. Then $x_i = y_i$ for all $i \geq M$, so

$$\begin{aligned} d(F(x), F(y)) &\leq \sum_{i=1}^\infty (K(i+1) - 1)(K(i+1) - K(i)) d(f_i(x_i), f_i(y_i)) \\ &\leq M^2 \sum_{i=1}^M d(f_i(x_i), f_i(y_i)) \\ &\leq M^2 \sum_{i=1}^M \rho_+^i(d(x_i, y_i)) \\ &\leq M^2 \sum_{i=1}^M \rho_+^M(d(x_i, y_i)) \\ &\leq M^3 \rho_+^M(d(x, y)) \end{aligned}$$

Since the last expression only depends on $d(x, y)$, we can make this the definition of ρ_+ . Since each f_i is injective, we can assume $\rho_-^i(t) \geq 1$ if $t \geq 1$. Let ρ_- be defined by

$$\rho_-(t) = \min \left(\sum_{i=1}^\infty i \cdot \rho_-^i(d(x_i, y_i)) : x, y \in \times_{i=1}^\infty G_i, d(x, y) \geq t \right)$$

Note that ρ_- is nondecreasing, and $\rho_-(d(x, y)) \leq d(F(x), F(y))$ for all $x, y \in \times_{i=1}^\infty G_i$. Suppose ρ_- is not proper, i.e. that there exists an integer m and a sequence of integers $n_1 < n_2 < \dots$ with $\rho_-(n_j) \leq m$ for every j . If $x, y \in \times_{i=1}^\infty G_i$ with $x_i \neq y_i$ for some $i > m$, then $d(x_i, y_i) \geq 1$, so $i \cdot \rho_-^i(d(x_i, y_i)) \geq i > m$. Hence for each n_j we can fix $x^j, y^j \in \times_{i=1}^\infty G_i$ such that x^j and y^j agree past m ,

$$d(x^j, y^j) \geq n_j$$

and the minimum in the definition of ρ_- is attained by x^j, y^j . Since the pairs x^j, y^j agree past m and $d(x^j, y^j) \rightarrow \infty$ as $j \rightarrow \infty$, for some $i < m$ the distances $d(x_i^j, y_i^j)$ increase without bound. But $\rho_-^i(d(x_i^j, y_i^j)) \leq \rho_-(d(x^j, y^j)) \leq m$ for each j , contradicting the properness of ρ_-^i . Hence ρ_- is proper, and ρ_-, ρ_+ witness that F is a coarse embedding. Hence $\times_{i=1}^\infty G_i$ has asymptotic property C. \square

4. HYPERBOLIC PROPERTY C

Buyalo and Schroeder introduced hyperbolic dimension in [BS], and Capadocia introduced weak hyperbolic dimension in [C]. We introduce infinite-dimensional versions of these properties.

Definition 4.1. A metric space X has *hyperbolic property C* if for every sequence R_0, R_1, \dots of positive reals there exists $n \geq 0$ and R_i -disjoint families \mathcal{U}_i , $i = 0, \dots, n$ such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a uniformly large scale doubling cover of X .

Definition 4.2. A metric space X has *weak hyperbolic property C* if for every sequence R_0, R_1, \dots of positive reals there exists $n \geq 0$ and R_i -disjoint families \mathcal{U}_i , $i = 0, \dots, n$ such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a weakly uniformly large scale doubling cover of X .

The following is immediate from the definitions.

Proposition 4.3. *Let X be a metric space. If X has asymptotic property C, then X has hyperbolic property C. If X has hyperbolic property C, then X has weak hyperbolic property C.*

Proposition 4.4. *If $\{X_\alpha : \alpha \in A\}$ is a family of metric spaces with finite asymptotic dimension uniformly, then $\{X_\alpha : \alpha \in A\}$ has sFDC.*

Proof. Suppose $\text{asdim } \{X_\alpha\} = n$, and consider a sequence of positive reals $R_0 < R_1 < \dots$. Then there exists $D > 0$ such that, for each α there exist $\mathcal{U}_0^\alpha, \mathcal{U}_1^\alpha, \dots, \mathcal{U}_n^\alpha$ such that \mathcal{U}_i^α is R_n -disjoint, $\text{mesh } \mathcal{U}_i^\alpha \leq D$ for each $i = 0, \dots, n$, and $\bigcup_{i=0}^n \mathcal{U}_i^\alpha$ is a cover of X_α . We define families \mathcal{V}_i^α for each α and \mathcal{V}_i inductively for $0 \leq i \leq n$. Let

$$\mathcal{V}_0^\alpha = \{X_\alpha \cap U : U \in \mathcal{U}_0^\alpha\}$$

and

$$\mathcal{V}_0 = \left(\bigcup_{\alpha \in A} \mathcal{V}_0^\alpha \right) \cup \{X_\alpha \setminus \bigcup \mathcal{V}_0^\alpha : \alpha \in A\}$$

For $0 < k < n$, having defined $\mathcal{V}_i^\alpha, \mathcal{V}_i$ for $i < k$, let

$$\mathcal{V}_k^\alpha = \{(X_\alpha \setminus \bigcup_{i=0}^{k-1} \mathcal{V}_i^\alpha) \cap U : U \in \mathcal{U}_k^\alpha\}$$

and

$$\mathcal{V}_k = \left(\bigcup_{\alpha \in A} \bigcup_{i=0}^k \mathcal{V}_i^\alpha \right) \cup \{X_\alpha \setminus \bigcup_{i=0}^k \mathcal{V}_i^\alpha : \alpha \in A\}$$

Finally, let

$$\mathcal{V}_n = \{(X_\alpha \setminus \bigcup_{i=0}^{n-1} \mathcal{V}_i^\alpha) \cap U : \alpha \in A, U \in \mathcal{U}_n^\alpha\} \cup \bigcup_{\alpha \in A} \bigcup_{i=0}^{n-1} \mathcal{V}_i^\alpha$$

Then \mathcal{V}_n is a subfamily of $\bigcup_{\alpha \in A} \bigcup_{i=0}^n \mathcal{U}_i^\alpha$, so \mathcal{V}_n is uniformly bounded. Also

$$\{X_\alpha\} \xrightarrow{R_0} \mathcal{V}_0 \xrightarrow{R_1} \mathcal{V}_1 \xrightarrow{R_2} \dots \xrightarrow{R_n} \mathcal{V}_n$$

since each \mathcal{U}_i^α is R_i -disjoint. We conclude that $\{X_\alpha\}$ has sFDC. \square

Theorem 4.5. *If X is a metric space with weak hyperbolic property C, then X has sFDC.*

Proof. Let X be a metric space with weak hyperbolic property C . Given a sequence $R_0 < R_1 < R_2 < \dots$, there exists $n \in \mathbb{N}$ and $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n$ with each \mathcal{U}_i an R_i -disjoint family of subsets of X , such that $\bigcup_{i=0}^n \mathcal{U}_i$ is a weakly large scale doubling cover of X . Let

$$\mathcal{V}_0 = \mathcal{U}_0 \cup \{X \setminus (\bigcup \mathcal{U}_0)\}$$

Clearly $\{X\} \xrightarrow{R_0} \mathcal{V}_0$. Next we R_1 -decompose \mathcal{V}_0 by intersecting $X \setminus (\bigcup \mathcal{U}_0)$ with the sets in \mathcal{U}_1 and so on. In the end we have \mathcal{V}_n a subfamily of $\bigcup_{i=0}^n \mathcal{U}_i$. Hence \mathcal{V}_n is weakly uniformly large scale doubling. By Proposition 4.5 in [NR], \mathcal{V}_n has finite asymptotic dimension uniformly. Hence by Proposition 4.4 there is $m \geq 0$ and a sequence of families $\mathcal{V}_{n+1}, \mathcal{V}_{n+2}, \dots, \mathcal{V}_{n+m}$ such that $\mathcal{V}_{n+i} \xrightarrow{R_{n+i+1}} \mathcal{V}_{n+i+1}$ for every $i < m$, and \mathcal{V}_{n+m} is uniformly bounded. Hence X has sFDC. \square

The above theorem and theorem 3.4 from [DZ] together yield the following:

Corollary 4.6. *If X is a metric space with weak hyperbolic property C , then X has Property A.*

REFERENCES

- [BBGRZ] T. Banach, B. Bokalo, I. Guran, T. Radul, M. Zarichnyi, *Problems from the Lviv topological seminar*, in E. Pearl (Ed.), *Open Problems in Topology II*, Elsevier (2007), 655-667.
- [BM] G. Bell, D. Moran, *On constructions preserving the asymptotic topology of metric spaces*, North Carolina Journal of Mathematics and Statistics, 1 (2015), (46-57).
- [BN] G. Bell, A. Nagórko, *On stability of asymptotic property C for products and some group extensions*.
- [BS] S. Buyalo and V. Schroeder, *Elements of asymptotic geometry*, EMS Monographs in Mathematics, European Mathematical Society, Zürich (2007).
- [C] C. Cappadocia, *Large scale dimension theory of metric spaces*, Ph.D. Thesis, McMaster University (2014).
- [D1] A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys 55 (2000), no. 6, 1085-1129.
- [D2] A. Dranishnikov, *On hypersphericity of manifolds with finite asymptotic dimension*, Trans. Amer. Math. Soc. 355 (2003), 155-167.
- [DZ] A. Dranishnikov, M. Zarichnyi, *Asymptotic dimension, decomposition complexity, and Haver's property C*, Topology Appl. 169 (2014), 99-107.
- [G] M. Gromov, *Asymptotic invariants of infinite groups*, in Geometric Group Theory, vol. 2, Cambridge University Press (1993).
- [GTU] E. Guentner, R. Tessera, G. Yu, *A notion of geometric complexity and its application to topological rigidity*, Invent. Math. 189 (2012), 315-357.
- [NR] A. Nicas, D. Rosenthal, *Hyperbolic dimension and decomposition complexity*, to appear in London Mathematical Society Lecture Notes Series.
- [S] J. Smith, *On asymptotic dimension of countable Abelian groups*, Topol. Appl. 153 (2006), 2047-2054.
- [Y] T. Yamauchi, *Asymptotic property C of the countable direct sum of the integers*, Topol. Appl. 184 (2015), 50-53.
- [Yu1] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math. (2) **147** (1998), no.2, 325-355.
- [Yu2] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), 201-240.